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Fractal Dimensions on a Nonlinear Discrete Model

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Abstract— In this paper we highlight three fractal dimensions: box-counting dimension, information dimension and correlation dimension to focus the geometric aspects of the attractor in a prominent one-dimensional nonlinear chaotic discrete model:

$$C(x) = m^2 x + 2mx^3 - x^5$$

where $x \in [0, 1.34]$ and $m \in [0.71, 0.82]$ is an adjustable parameter Our system undergoes a period-doubling bifurcation route to chaos and consequently finding the accumulation point we obtain some illuminating results.

Key Words- Fractal dimension, Box-counting dimension, Information dimension, Correlation dimension, Attractor, Chaotic, Period-doubling bifurcation, Accumulation point.

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1 INTRODUCTION

Over the past few decades, distinguishing deterministic chaos from noise has become an important problem in many diverse fields. This is due, in part, to the availability of numerical algorithms for quantifying chaos using experimental time series. In particular, methods exist for calculating correlation dimension, Lyapunov characteristic exponents, etc. Dissipative dynamical systems which exhibit chaotic behavior often have an attractor in phase space which is strange. While the main purpose of Lyapunov exponents is to characterize the dynamical properties of trajectories on attractors, the fractal dimensions focuses on the geometry of the attractor. Chaotic dynamical systems exhibit trajectories in their phase space that converge to a strange attractor. The fractal dimension of this attractor counts the effective number of degrees of freedom in the dynamical system and thus quantifies its complexity. To compute dimensions of strange attractors researchers have designed some mechanisms, for example information dimension, correlation dimension etc, to reflect not only the fractal geometry of the underlying objects but also the dynamics which take place in them [1, 3, 7, 9, 11, 13, 14].

Our aim in this paper is to estimate some fractal dimensions: box-counting dimension, information dimension and correlation dimension as measure of chaos for the one-dimensional nonlinear chaotic discrete model:

$$C(x) = m^2 x + 2mx^3 - x^5 \tag{1}$$

where $x \in [0, 1.34]$ and the control parameter $m \in [0.71, 0.82]$

We now highlight some useful concepts which are absolutely useful for our purpose.

1.1 Discrete dynamical systems

Any $C^k (k \geq 1)$ map $E : U \to \Re^n$ on the open set $U \subset R^n$ defines an n-dimensional **discrete-time** (autonomous) smooth dynamical system by the state equation

$$\bar{x}_{t+1} = E(\bar{x}_t), t = 1,2,3,..... \tag{2}$$

where $\bar{x}_t \in \Re^n$ is the state of the system at time t and E maps $\bar{x}_t$ to the next state $\bar{x}_{t+1}$. Starting with an initial data $\bar{x}_0$, repeated applications (iterates) of E generate a discrete set of points (the orbits) $\{E^t(\bar{x}_0) : t = 0,1,2,3,.....\}$, where $E^t(\bar{x}) = \underbrace{E \circ E \circ ... \circ E(\bar{x})}_{t \text{ times}}$ [12].

1.2 Definition: The closed set $\boldsymbol{A} \in \Re^n$ is called the attractor of the system (2) if (i) there exists an open set $\boldsymbol{A}_0 \supset \boldsymbol{A}$ such that all trajectories $\bar{x}_t$ of system (2) beginning in $\boldsymbol{A}_0$ are definite for all $t \geq 0$ and tend to $\boldsymbol{A}$ for $t \to \infty$, that is, $\text{dist}(\bar{x}_t, \boldsymbol{A}) \to 0$ for $t \to \infty$, if $\bar{x}_0 \in \boldsymbol{A}_0$, where

$$\text{dist}(\bar{x}, \boldsymbol{A}) = \inf_{\bar{y} \in \boldsymbol{A}} \|\bar{x} - \bar{y}\|$$

is the distance from the point $\bar{x}$ to the set $\boldsymbol{A}$, and (ii) no eigensubset of $\boldsymbol{A}$ has this property.

1.3 Box-counting dimension

There are several equivalent definitions of this dimension. One possibility is to partition the phase space of the system concerned by hyper-cubes of a side length ε. Then we call $N_\varepsilon(\boldsymbol{A})$ the number of cells, which are intersected by the attractor $\boldsymbol{A}$. Then box-counting dimension D_b is defined as

$$D_b = \lim_{\varepsilon \to 0} \left[-\frac{\log N_\varepsilon(\boldsymbol{A})}{\log(\varepsilon)}. \right] \tag{3}$$

This is a property of the set $\boldsymbol{A}$ only. It is invariant with respect to smooth invertible transfo-
rmations of the phase space.

1.4 The information dimension [2]

As with the box-counting dimension, the attractor is covered with hypercubes of side length ε. This time, however, instead of simply counting each cube which contains part of the attractor, we want to know how much of the attractor is contained within each cube. This measure seeks to account for differences in the distribution density of points covering the attractor, and is defined as

$$D_I = \lim_{\varepsilon \to 0} \frac{I(\varepsilon)}{\log(1/\varepsilon)} \tag{4}$$

where $I(\varepsilon)$ is given by Shannon's entropy formula,

$$I(\varepsilon) = -\sum_{i=1}^{N} P_i \log(P_i) \tag{5}$$

where P_i is the probability of part of the attractor occurring within the ith hypercube of side length ε. For the special case of an attractor with an even distribution of points, an identical probability, $P_i = 1/N$, is associated with every box. Hence, $I(\varepsilon) = \text{In}(N)$. Consequently, $B_b = B_I$. Thus, D_b simply counts all hypercubes containing parts of the attractor, whereas D_I asks how much of the attractor is within each hypercube and correspondingly weights its count.

1.5 Correlation dimension [2]

A practical method to estimate the fractal dimension of the phase space goes through the calculation of the correlation dimension, D_c which is computationally efficient and relatively fast when implemented as an algorithm for dimension estimation. This technique is based on the behavior of the so-called **correlation sum**, C_ε where

$$C_\varepsilon = \frac{1}{N(N-1)} \sum_{i=1}^{N} \sum_{j=1;\ j\neq i}^{N} \theta\left(\varepsilon - \left|X_i - X_j\right|\right) \tag{6}$$

that is , an estimate of the probability that the two points X_i, X_j on the attractor lie less than a distance ε from each other. Here, θ is the Heaviside step function.

The correlation sum scales with the hypersphere radius according to a power law of the form

$$C_\varepsilon \propto \varepsilon^{D_c}$$

where the exponent, D_c, is the correlation dimension. Hence, by examining the attractor in the method cited above, for different hypersphere radii, D_c is obtained from the slope of the scaling region of a $\log(\varepsilon) - \log(C_\varepsilon)$ plot.

1.6 Generalized dimension

All of above cited measures of the “dimension” are averages over the attractor. Stating just an average value does not acknowledge all of the complexity of the attractor’s geometry. To provide more detailed information about the geometry of the attractor we discuss the generalized dimensions [8, 10, 14]. It is defined as follows:

The generalized (box-counting) fractal dimensions D_q, known as Renyi dimensions, where $q \in \Re$, are defined by

$$D_q = \lim_{\varepsilon \to 0} \frac{1}{q-1} \frac{\log \sum_{i=1}^{N(\varepsilon)} p_i^q(\varepsilon)}{\log \varepsilon} \tag{7}$$

where the index i labels the individuals boxes of size ε and $p_i(\varepsilon)$ denotes the relative weight of the i th box or the probability of the object lying in the box and is defined as $p_i(\varepsilon) = N_i(\varepsilon)/N$, where $N_i(\varepsilon)$ is the weight (number of iterated points) of the i th box and N is the total weight of the object, i.e. N is total number of iterated points.

The factor of $q-1$ has been included in the denominator so that for $q = 0$, we have

$$D_0 = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{-\log(\varepsilon)}$$

which is same as the box-counting dimension D_b. When $q = 1$, L' Hospital’s rule can be applied to give

$$D_I = \lim_{\varepsilon \to 0} \frac{\sum_{i=1}^{N} p_i \log(p_i)}{-\log(\varepsilon)}$$

which is known as the **information dimension**. The quantity D_c is known as the **correlation dimension** and indicates the correlation between pairs of points in each box. The generalized dimensions $D_3, D_4,...$ are associated with correlations between triples, quadruples, etc., of points in each box.

Again, generalized correlation sum is given as follows:

$$C_q(N,\varepsilon) = \left[\frac{1}{N-1} \sum_{i=1}^{N} \left[\frac{1}{N-1} \sum_{j=1,\, j\neq i}^{N} \theta(\varepsilon - \left\|X_i - X_j\right\|) \right]^{q-1} \right]^{\frac{1}{q-1}}$$

where

$$\theta\left(\varepsilon - \left\|X_i - X_j\right\|\right) = \begin{cases} 1, & if \left\|X_i - X_j\right\| < \varepsilon \\ 0, & if \left\|X_i - X_j\right\| \geq \varepsilon \end{cases}$$

The generalized dimensions D_q, in terms of the generalized correlation sum

$$D_q = \lim_{\varepsilon \to 0} \frac{1}{q-1} \frac{\log C_q(N,\varepsilon)}{\log \varepsilon}$$

Moreover,

$$\lim_{N \to \infty} C_q(N,\varepsilon) = C_q(\varepsilon)$$

and the generalized dimensions D_q, in terms of the generalized correlation sum

$$D_q = \lim_{\varepsilon \to 0} \frac{1}{q-1} \frac{\log C_q(\varepsilon)}{\log \varepsilon} \tag{8}$$

Finding $C_q(N,\varepsilon)$, we may have the value of $C_q(\varepsilon)$ and in the long run we can have C_q for a particular value of q.

Armed with all these ideas and concepts, we now proceed to concentrate to our main aim and objectives.

2. Calculation of fractal dimensions near the accumulation point

Using the Feigenbaum theory we have the following period-doubling cascade table [4-6]:

TABLE 1

One of Periodic points	Bifurcation values
$x_1 = 1.089972242373...$	$m_1 = 0.767178057711...$
$x_2 = 1.191337056772...$	$m_2 = 0.786658842890...$
$x_3 = 1.045288314133...$	$m_3 = 0.791113613628...$
$x_4 = 1.028375916681...$	$m_4 = 0.792077950474...$
$x_5 = 1.035431857429...$	$m_5 = 0.792285006780...$
$x_6 = 1.038231474079...$	$m_6 = 0.792329374414...$
$x_7 = 1.038805426726...$	$m_7 = 0.792338877675...$
$x_8 = 1.033471877356...$	$m_8 = 0.792340913030...$
$x_9 = 1.038030653420...$	$m_9 = 0.792341348943...$
……………………………….	………………………………

Our system undergoes a period-doubling bifurcation and so let $\{m_n\}$ be the sequence of bifurcation points. Using Feigenbaum δ, if the first and second bifurcation points m_1 and m_2 are known then the

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third point m_3 can be predicted as:

$$m_3 \approx \frac{m_2 - m_1}{\delta} + m_2 \qquad (9)$$

Of course, the first two period-doublings produces no guarantee that a third will occur, but if it does occur, the equation (9) gives us a reasonable prediction of the parameter value near which we should look to see the transition. Similarly knowing m_2 and m_3, we can a reasonable prediction for m_4 as

$$m_4 \approx \frac{m_3 - m_2}{\delta} + m_3 \qquad (10)$$

which implies

$$m_4 \approx (m_2 - m_1)\left(\frac{1}{\delta} + \frac{1}{\delta^2}\right) + m_2 \qquad (11)$$

Repeating this argument, we just get more terms in the sum involving powers of $(1/\delta)$ and recognizing this sum as a geometric series, we can sum the series to obtain the following result [10]:

$$m_\infty \approx (m_2 - m_1)\frac{1}{\delta - 1} + m_2 \qquad (12)$$

However, this expression is exact when the bifurcation ratio

$$\delta_n = \frac{m_n - m_{n-1}}{m_{n+1} - m_n},$$

is equal for all values of *n*. In fact $\{\delta_n\}$ converges as $n \to \infty$, that is, $\lim_{n\to\infty} \delta_n = \delta$. So we consider the sequence

$$\{m_{\infty,n}\},\ m_{\infty,n} \approx (m_n - m_{n-1})\frac{1}{\delta - 1} + m_n$$

where m_n are the experimental value of bifurcation points.

Evidently, $\lim_{n\to\infty} m_{\infty,n} = m_\infty$.

Using the experimental bifurcation points of the Table 1, the sequence $\{m_{\infty,n}\}$ is calculated for some values of n as $\{m_{\infty,1}, m_{\infty,2}, m_{\infty,3},......, m_{\infty,8},......\}$ where

$m_{\infty,1}$ = *0.7919681137477003..*

$m_{\infty,2}$ = *0.7923277117374795..*

$m_{\infty,3}$ = *0.7923407697323835..*

$m_{\infty,4}$ = *0.7923414376682444..*

$m_{\infty,5}$ = *0.7923414663181022..*

$m_{\infty,6}$ = *0.7923414676828573..*

$m_{\infty,7}$ = *0.7923414677433182..*

$m_{\infty,8}$ = *0.7923414677462293..*

… … …

… … …

The above sequence converges to the value *0.79234146774…..*, which is the required **accumulation point**.

1. Determination of fractal dimensions of our model

To measure the dimension of the attractor near the onset of chaos, that is, near the accumulation point *A=0.79234146774…..*, we calculate box-counting dimension, information and correlation dimension with the help of the generalized dimensions described as above.

3.1 Correlation dimension

We calculate $C_q(N,\varepsilon)$ for as large as it is suitable for our computer program given in the appendix and assume it as $C_q(\varepsilon)$ for *q* = *2*. The part of the plotted points $(\log\varepsilon,\ \log C_q(\varepsilon))$, as shown in the Box 1, which follows equation (8), is taken. The gradient of the best-fit line in that scaling region is D_q

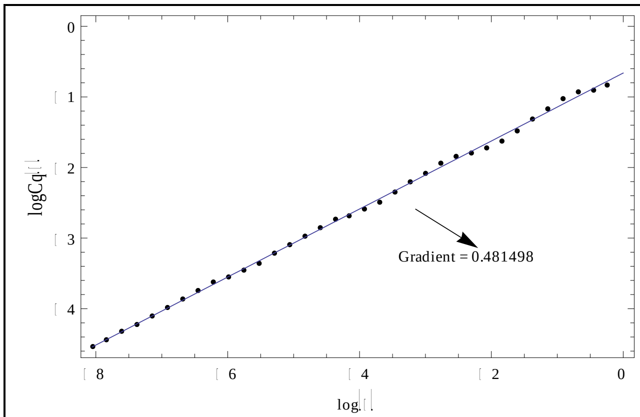

Fig. 1. A plot of for the map (1) trajectories to find correlation dimension.

The gradient of the above points when fitted with a straight line by least square method is 0.481498 with a mean deviation of 0.100789. The data is obtained from 30000 iterated points at the parameter value *0.79234146774...*

Box 1

{-8.059048,-4.527607},{-7.828789,-4.438109},{-7.598531,-4.331872},{-7.368272,4.228933},{-7.138014,-4.094042},{-6.907755,-3.977210},{-6.677497,-3.866198},{-6.447238,-3.754364},{-6.216980,-3.634642},{-5.986721,-3.558636},{-5.756463,-3.453442},{-5.526204,-3.353792},{-5.295946,-3.211312},{-5.065687,-3.092915},{-4.835429,-2.971285},{-4.605170,-2.850103},{-4.374912,-2.738767},{-4.144653,-2.676009},{-3.914395,-2.582476},{-3.684136,-2.482786},{-3.453878,-2.336109},{-3.223619,-2.207845},{-2.993361,-2.075412},{-2.763102,-1.941794},{-2.532844,-1.835705},{-2.302585,-1.792740},{-2.072327,-1.722738},{-1.842068,-1.620624},{-1.611810,-1.481783},{-1.381551,-1.308827},{-1.151293,-1.173209},{-0.921034,-1.023145},{-0.690776,-0.928256},{-0.460517,-0.905486},{-0.230259,-0.842827}

3.2 Information dimension

In the equation (1.8) if we take *q* very near to 1, say *q* = *1.00000000001* then that gives the information dimension. At the parameter value *0.79234146774…* with 30000 iterations in the attractor, the gradient of the graph when fitted a straight line in the scaling region is 0.498484 with a mean deviation 0.0780595 which may be called as the information dimension.

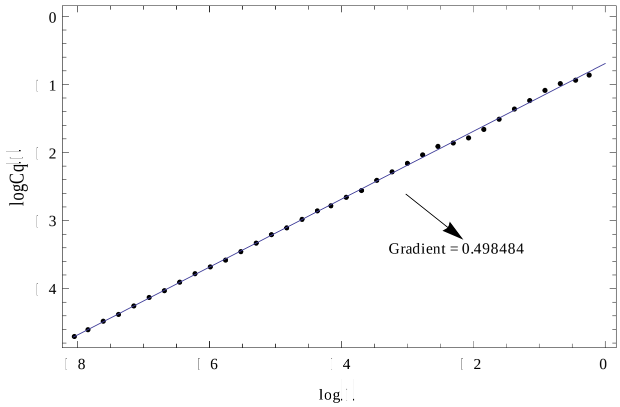

Fig. 2 A plot of for the map (1) trajectories to find information dimension.

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3.3 Box-counting dimension

If we take $q = 0$ in the (1.8), we get the box-counting dimension as 0.508708 with a mean deviation 0.101432 at the parameter *0.79234146774* with 30000 iterations.

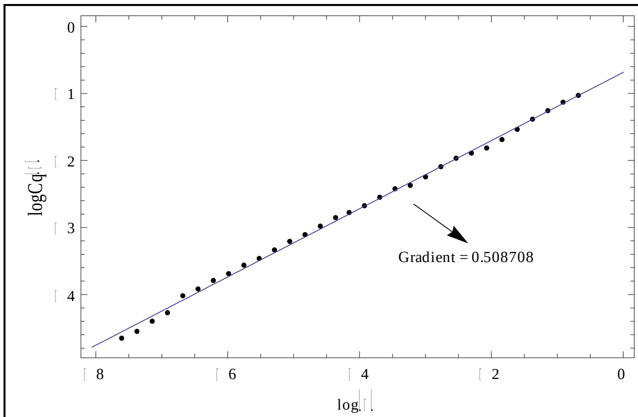

Fig. 3 A plot of for the map (1) trajectories to find box-counting dimension

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3.4 Conclusion

(i) We can conjecture that our above method can be applied for determining the dimensions of any higher order chaotic maps.

(ii) One natural question with a dynamical system is: how chaotic is a system's chaotic behavior? To give a quantitative answer to that question, we know that the dimension theory plays a crucial role. Why is dimensionality important? One possible answer is that the dimensionality of the state space is closely related to dynamics. The dimensionality is important in determining the range of possible dynamical behavior. So our above discussion on various dimensions sheds light on beautiful dynamical behaviors of the systems.

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